ISRAEL JOURNAL OF MATHEMATICS 114 (1999), 347–357

SPLITTING $\mathcal{P}_\kappa\lambda$ INTO MAXIMALLY MANY STATIONARY SETS

BY

MASAHIRO SHIOYA*

Institute of Mathematics, University of Tsukuba
Tsukuba 305-8571, Japan
e-mail: shioya@math.tsukuba.ac.jp

ABSTRACT

Let $\kappa > \omega$ be a regular cardinal and $\lambda > \kappa$ a cardinal. We show that $\mathcal{P}_\kappa\lambda$ splits into λ^ω stationary sets.

0. Introduction

Let $\kappa > \omega$ be a regular cardinal and $\lambda > \kappa$ a cardinal. Solovay's classical result for κ [So] led Menas [Me] to conjecture that a stationary subset of $\mathcal{P}_\kappa\lambda$ would split into $\lambda^{<\kappa}$ stationary sets. Unfortunately his conjecture fails when $2^{<\kappa} > \kappa^+$: While $\mathcal{P}_\kappa\kappa^+$ carries a stationary set of size κ^+ (see [BT]), the conjecture implies that the size is $(\kappa^+)^{<\kappa}$ as well.

What about splitting a stationary set S into $\min\{|S \cap C| : C \text{ is club}\}$ many sets? Gitik's answer [G] was again negative: Relative to supercompactness, it is consistent that some stationary subset of $\mathcal{P}_\kappa\kappa^+$ splits into at most κ stationary sets.

Now it seems natural to ask the same question as above for a canonical stationary set. Let us concentrate on the case where the canonical set is $\mathcal{P}_\kappa\lambda$ itself. When $\kappa = \omega_1$, we have a satisfactory answer by the works of Baumgartner–Taylor [BT] (the case $\lambda \leq 2^\omega$) and Donder–Matet [DM] (otherwise): $\mathcal{P}_{\omega_1}\lambda$ splits into λ^ω stationary sets. In fact the latter proved the diamond principle for $\mathcal{P}_\kappa\lambda$ when $\lambda > 2^{<\kappa}$.

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* Part of this work was done during the author's stay at Boston University as one of the Japanese Overseas Research Fellows. He gratefully acknowledges Professor Akihiro Kanamori's hospitality. He also wishes to thank members of the set theory seminar at Waseda University for their interest at the early stage.
Received December 28, 1997

In this paper we are mainly concerned with the general type of result as follows (see [Ka]): $\mathcal{P}_\kappa\lambda$ splits into λ stationary sets. As suggested above, we should first measure the minimum size of a club subset of $\mathcal{P}_\kappa\lambda$. Elaborating his earlier result [BT], Baumgartner [B] has already shown that it is at least λ^ω. This and the following result of Magidor [Mag] imply that λ^ω is the critical number for our specific splitting problem: If there is no ω_1-Erdős cardinal in the Dodd–Jensen core model, $\mathcal{P}_\kappa\lambda$ carries a club set of size λ^ω when $\operatorname{cf}\lambda \geq \kappa$, and of size $\max\{\lambda^\omega, \lambda^+\}$ otherwise.

Unifying three of the results above, we establish the desired splitting:

THEOREM 1: *$\mathcal{P}_\kappa\lambda$ splits into λ^ω stationary sets.*

We also realize the splitting suggested in the latter case of Magidor's theorem:

THEOREM 2: *$\mathcal{P}_\kappa\lambda$ splits into λ^+ stationary sets when $\operatorname{cf}\lambda < \kappa$.*

1. Preliminaries

Our notation should be standard. Kanamori's book [Ka] is an excellent source for background material. We reserve κ for a regular cardinal $> \omega$, λ for a cardinal $> \kappa$ and μ, ν for a cardinal $\geq \omega$. When $\mu < \kappa$ is regular, S^μ_κ (resp. $S^{<\mu}_\kappa$, $S^{\geq\mu}_\kappa$) denotes the set of limit ordinals $< \kappa$ of cofinality μ (resp. $< \mu$, $\geq \mu$). For a set X of ordinals let $\lim X$ be the set $\{\gamma < \sup X : \sup(X \cap \gamma) = \gamma > 0\}$ of limit points of X and $\operatorname{cl}_f X$ the closure of X under $f : \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$, i.e. the minimal set $Y \supset X$ with $\bigcup f``Y^{<\omega} \subset Y$. Unless otherwise stated, we understand that a set of ordinals is listed in increasing order and a splitting of a stationary set is mutually disjoint.

Thoughout the paper we freely use Solovay's theorem [So] mentioned earlier:

THEOREM: *A stationary subset of κ splits into κ stationary sets.*

We need a version of Shelah's club guessing sequence (see [Ko]). Let us sketch a proof due to Hirata [H]:

THEOREM: *Let $\mu < \kappa < \lambda$ be all regular and $S \subset S^\mu_\lambda \cap \lim S^{\geq\kappa}_\lambda$ stationary. Then there is a sequence $\langle c_\gamma : \gamma \in S\rangle$ such that $c_\gamma \subset S^{\geq\kappa}_\lambda$ is unbounded in γ and of order type μ for any $\gamma \in S$ and $\{\gamma \in S : c_\gamma \subset C\}$ is stationary for any club set $C \subset \lambda$.*

Proof: First for $\beta \in \lim\lambda$ fix an unbounded set $d_\beta \subset \beta$ of order type $\operatorname{cf}\beta$. For $\gamma \in S$ and a club set $D \subset \lim\lambda$ set $x^D_\gamma = \bigcup_{n<\omega} x^D_{\gamma,n} - \{0\}$, where $x^D_{\gamma,n}$ is defined inductively by

$$x^D_{\gamma,0} = \{\sup(D \cap \alpha) : \alpha \in d_\gamma\}$$

and

$$x^D_{\gamma,n+1} = \{\sup(D\cap\alpha) : \exists\beta \in x^D_{\gamma,n}\cap S^{<\kappa}_\lambda(\alpha\in d_\beta)\}.$$

Note that $x^D_\gamma \subset D$ since D is closed, and $|x^D_{\gamma,n}| < \kappa$ by induction on $n<\omega$. First we find a club set $D\subset\lambda$ such that $\{\gamma\in S : x^D_\gamma \subset C\}$ is stationary for any club set $C\subset\lambda$.

Otherwise we would have inductively a descending sequence $\langle C_\xi : \xi<\kappa\rangle$ of club subsets of $\lim\lambda$ such that $C_{\xi+1}\cap\{\gamma\in S : x^{C_\xi}_\gamma \subset C_{\xi+1}\} = \emptyset$ for any $\xi<\kappa$. Fix $\gamma\in S\cap\bigcap_{\xi<\kappa} C_\xi$. Then we have inductively $\{\xi_n : n<\omega\}\subset\kappa$ such that $x^{C_\xi}_{\gamma,n} = x^{C_{\xi_n}}_{\gamma,n}$ for any $\xi_n\le\xi<\kappa$, since the map $\xi\mapsto\sup(C_\xi\cap\alpha)$ is decreasing for any $\alpha<\lambda$ and $|x^{C_{\xi_n}}_{\gamma,n}|<\kappa$ by the note above. Set $\xi=\sup_{n<\omega}\xi_n<\kappa$. Then $x^{C_\xi}_\gamma = x^{C_{\xi+1}}_\gamma \subset C_{\xi+1}$ by the note above. This contradicts

$$C_{\xi+1}\cap\{\gamma\in S : x^{C_\xi}_\gamma\subset C_{\xi+1}\}=\emptyset.$$

Now fix a club set $D\subset\lambda$ as above. Then $S^* = \{\gamma\in S\cap\lim D : x^D_\gamma\subset\lim D\}$ is stationary by the claim above. Fix $\gamma\in S^*$. We have $x^D_\gamma - \lim x^D_\gamma \subset S^{\ge\kappa}_\lambda$, since $\beta\in x^D_{\gamma,n}\cap S^{<\kappa}_\lambda$ implies $\beta\in\lim x^D_{\gamma,n+1}$ by $\beta\in\lim D$. Also $x^D_\gamma - \lim x^D_\gamma$ is unbounded in γ, since $x^D_{\gamma,0}$ is unbounded in γ by $\gamma\in\lim D$.

Finally we get the desired sequence by taking an unbounded subset of $x^D_\gamma - \lim x^D_\gamma$ of order type μ as c_γ for $\gamma\in S^*$. ∎

In fact we use only the sequence of the form $\langle c_\gamma : \gamma\in S^\omega_\lambda\rangle$ and do not appeal to the clause $c_\gamma\subset S^{\ge\kappa}_\lambda$. The second result we quote from Shelah's pcf theory is a scale on a singular cardinal [Sh] (see also [BMag]):

THEOREM: *Let λ be singular. Then there are an unbounded set $\{\lambda_\xi : \xi<\operatorname{cf}\lambda\}\subset\lambda$ of regular cardinals and $\{f_\gamma : \gamma<\lambda^+\}\subset\prod_{\xi<\operatorname{cf}\lambda}\lambda_\xi$ such that $f_\beta\le^* f_\gamma$ for any $\beta<\gamma<\lambda^+$ and for any $g\in\prod_{\xi<\operatorname{cf}\lambda}\lambda_\xi$ there is $\gamma<\lambda^+$ with $g\le^* f_\gamma$.*

Here $\le^*$ denotes the eventual dominance: $f\le^* g$ iff $\{\xi<\operatorname{cf}\lambda\colon f(\xi)\le g(\xi)\}$ is cobounded. The later development of the theory as presented in [Ko] yields a more transparent proof of this deep result.

2. Main theorems

This section is devoted to establishing Theorems 1 and 2.

Our proof of Theorem 1 consists of two major parts. For the first part we are strongly indebted to Todorčević [T2], who reproved Gitik's answer [G] to Abraham's question [AS] and claimed that his method would yield the Baumgartner–Taylor result as well via the following: Let $\langle c_\gamma : \gamma\in S^\omega_{\omega_2}\rangle$ be a club guessing sequence with $c_\gamma = \{\gamma_n : n < \omega\}$. Then $\{x \in \mathcal{P}_{\omega_1}\omega_2 : \exists\gamma \in S^\omega_{\omega_2}(\sup x = \gamma \wedge \{n < \omega\colon x \cap (\gamma_{n+1} - \gamma_n) \neq \emptyset\} = r)\}$ is stationary for any $r \in [\omega]^\omega$.

Let λ be regular. We endow $[\lambda]^{<\omega}$ with the tree ordering $\leq\ = \{(a, b) : a$ is an initial segment of $b\}$. Let T be a subtree of $[\lambda]^{<\omega}$, i.e. a subset of $[\lambda]^{<\omega}$ closed under initial segments. Set $[T] = \{B \in [\lambda]^\omega : \forall\beta \in B(B \cap \beta \in T)\}$, the set of infinite branches through T, and $T^a = \{b \in [\lambda]^{<\omega} : a \leq a \cup b \in T\}$, the tree above $a \in [\lambda]^{<\omega}$. We call $T \neq \emptyset$ stationary if the set of immediate successors of $a \in T$, $\mathrm{suc}_T(a) = \{\alpha < \lambda : a \leq a \cup \{\alpha\} \in T\}$ is always stationary, and $g\colon T \to \lambda$ regressive when $g(a) \leq g(b) \in \min b \cup \{0\}$ for any $a \leq b \in T$.

Let us start with a tree version of the regressive function lemma:

LEMMA: *Let $g : T \to \lambda$ be regressive with T a stationary subtree of $[S^\kappa_\lambda]^{<\omega}$. Then for some stationary subtree T^* of T, $g``T^*$ is bounded in λ.*

Proof: For $\gamma < \lambda$ set $T_\gamma = \{a \in T : g(a) < \gamma\}$, a subtree of T by order preservation of g. First we find $\gamma < \lambda$ with $[T_\gamma] \cap [C]^\omega \neq \emptyset$ for any club set $C \subset \lambda$.

Suppose to the contrary that for $\gamma < \lambda$ we have a club set $C_\gamma \subset \lambda$ with $[T_\gamma] \cap [C_\gamma]^\omega = \emptyset$. Take inductively $B \in [T] \cap [\Delta_{\gamma<\lambda} C_\gamma]^\omega$ by the stationarity of T. Take $\alpha < \min B$ with $B \in [T_\alpha]$ by $\mathrm{cf} \min B = \kappa > \omega$ and the regressiveness of g. Then $B \in [C_\alpha]^\omega$ by $B \in [\Delta_{\gamma<\lambda} C_\gamma]^\omega$. This contradicts $[T_\alpha] \cap [C_\alpha]^\omega = \emptyset$ by the choice of C_α.

Fix $\gamma < \lambda$ as above. Set $T^* = \{a \in T_\gamma : \forall b \leq a \forall C \subset \lambda \text{ club } ([{T_\gamma}^b] \cap [C]^\omega \neq \emptyset)\}$, a subtree of T. Note that $\emptyset \in T^*$ by the choice of γ. We claim that T^* is stationary as desired.

Suppose to the contrary that $D \cap \mathrm{suc}_{T^*}(a) = \emptyset$ for some $a \in T^*$ and some club set $D \subset \lambda$. Then for $\alpha \in D$ we have a club set $C_\alpha \subset \lambda$ with $[{T_\gamma}^{a\cup\{\alpha\}}] \cap [C_\alpha]^\omega = \emptyset$ by $a \in T^*$ and $a \cup \{\alpha\} \notin T^*$. Thus $C = D \cap \Delta_{\alpha\in D} C_\alpha$ is club in λ. Take $B \in [{T_\gamma}^a] \cap [C]^\omega$ by $a \in T^*$. Set $\beta = \min B$. Then $B - \{\beta\} \in [{T_\gamma}^{a\cup\{\beta\}}]$ by $B \in [{T_\gamma}^a]$, and $B - \{\beta\} \in [C_\beta]^\omega$ by $B \in [C]^\omega$. This contradicts $[{T_\gamma}^{a\cup\{\beta\}}] \cap [C_\beta]^\omega = \emptyset$ by $\beta \in D$ and the choice of C_β. ∎

For the following lemma we fix a club guessing sequence $\langle c_\gamma : \gamma \in S^\omega_\lambda\rangle$ with $c_\gamma = \{\gamma_n : n < \omega\}$.

MAIN LEMMA 1: *Let $S_n \subset S^\kappa_\lambda$ be stationary for $n < \omega$. Then*

$$\{x \in \mathcal{P}_\kappa\lambda : \exists\gamma \in S^\omega_\lambda(\sup x = \gamma \wedge \forall n < \omega(\min(x - \gamma_n) \in S_n))\}$$

is stationary.

Proof: Fix $f\colon \lambda^{<\omega} \to P_\kappa\lambda$. Set

$$T = \{a : \forall n < |a|(\text{the } n\text{th element of } a \text{ is in } S_n)\},$$

a stationary subtree of $[S^\kappa_\lambda]^{<\omega}$. We build inductively a stationary subtree T_n of T and $h_n\colon T_n \cap [\lambda]^n \to \lambda$ so that $T_{n+1} \subset T_n$, $T_{n+1} \cap [\lambda]^n = T_n \cap [\lambda]^n$ and $\mathrm{cl}_f(a \cup B) \cap \min B \subset h_n(a)$ for any $a \in T_{n+1} \cap [\lambda]^n$ and $B \in [{T_{n+1}}^a]$.

First set $T_0 = T$. Next suppose that T_n is defined. Fix $a \in T_n \cap [\lambda]^n$. Then the map $g_a : b \mapsto \sup(\mathrm{cl}_f(a \cup b) \cap \min b)$ is regressive on ${T_n}^a$ by $\mathrm{cf} \min b = \kappa$. By the lemma above we have a stationary subtree T_a of ${T_n}^a$ and $h_n(a) < \lambda$ with $g_a``T_a \subset h_n(a)$. Then $T_{n+1} = (T_n \cap [\lambda]^{<n}) \cup \{a \cup b : a \in T_n \cap [\lambda]^n \wedge b \in T_a\}$ is the desired stationary subtree of T_n: Fix $a \in T_{n+1} \cap [\lambda]^n$ and $B \in [{T_{n+1}}^a]$. Then $\mathrm{cl}_f(a \cup B) \cap \min B = \bigcup_{\beta \in B} \mathrm{cl}_f(a \cup (B \cap \beta)) \cap \min B \subset \bigcup_{\beta \in B} g_a(B \cap \beta) \subset h_n(a)$.

Now set $T^* = \bigcap_{n<\omega} T_n$, a stationary subtree of T, and $h = \bigcup_{n<\omega} h_n\colon T^* \to \lambda$. Then $C = \{\gamma < \lambda : \mathrm{cl}_f \gamma = \gamma \wedge \forall a \in T^* \cap [\gamma]^{<\omega}(h(a) < \gamma \wedge \gamma \in \lim \mathrm{suc}_{T^*}(a))\}$ contains a club set. Fix $\gamma \in S^\omega_\lambda \cap C$ with $c_\gamma = \{\gamma_n : n < \omega\} \subset C$. Take inductively $B = \{\beta_n : n < \omega\} \in [T^*]$ so that $\gamma_n < \beta_n < \gamma_{n+1}$ by $\gamma_{n+1} \in C$ and the inductive hypothesis $\{\beta_i : i < n\} \in T^* \cap [\gamma_n]^{<\omega}$. Then $\mathrm{cl}_f B$ is as desired: First we have $\sup \mathrm{cl}_f B = \gamma$, since $\sup B = \gamma$ and $\mathrm{cl}_f B \subset \mathrm{cl}_f \gamma = \gamma$ by $\gamma \in C$. Next $\min(\mathrm{cl}_f B - \gamma_n) = \beta_n$, since $\mathrm{cl}_f B \cap \beta_n \subset h_n(B \cap \beta_n) = h(B \cap \beta_n) < \gamma_n$ by $\gamma_n \in C$ and $B \cap \beta_n \in T^* \cap [\gamma_n]^{<\omega}$. ∎

The following lemma is due to Foreman–Magidor [FM], who introduce the notion of mutual stationarity and show that the club filter on $\mathcal{P}_{\omega_1}\lambda$ is not $\lambda^{\mathrm{cf}\,\lambda}$-saturated when λ is singular.

Let $\mathrm{cf}\,\lambda = \omega$ and $\{\lambda_n : n < \omega\} = \{\kappa_i : i < \omega\} \subset \lambda$ an unbounded set of regular cardinals $> \kappa$ such that $\lambda_n < \lambda_{n+1}$ and $\{i < \omega : \kappa_i = \lambda_n\}$ is infinite for any $n < \omega$. Let W be the tree $\bigcup_{m<\omega} \prod_{i<m} \kappa_i$ ordered by inclusion. For a subtree T of W set $[T] = \{B \in \prod_{i<\omega} \kappa_i : \forall m < \omega(B|m \in T)\}$, the set of infinite branches through T, and $\mathrm{suc}_T(s) = \{\alpha : s * \langle\alpha\rangle \in T\}$, the set of immediate successors of $s \in T$.

Main Lemma 2: *Let $S_n \subset S^\omega_{\lambda_n}$ be stationary for $n < \omega$. Then*

$$\{x \in \mathcal{P}_\kappa\lambda : \forall n < \omega(\sup(x \cap \lambda_n) \in S_n)\}$$

is stationary.

Proof: Fix $f\colon \lambda^{<\omega} \to P_\kappa\lambda$. We build inductively a subtree T_n of W so that $T_{n+1} \subset T_n$, $\sup(\mathrm{cl}_f \operatorname{ran} B \cap \lambda_{n-1}) \in S_{n-1}$ for any $B \in [T_n]$, and for any $s \in T_n$, $\mathrm{suc}_{T_n}(s)$ is a singleton if $\kappa_{|s|} < \lambda_n$, and is unbounded in $\kappa_{|s|}$ otherwise.

First set $T_0 = W$. Next suppose that T_n is defined. For $\gamma < \lambda_n$ we call a subtree $U \neq \emptyset$ of W cobounded below γ if for any $s \in U$, $\mathrm{suc}_U(s)$ is $\kappa_{|s|}$ if $\kappa_{|s|} < \lambda_n$, and is cobounded in γ (resp. $\kappa_{|s|}$) if $\kappa_{|s|} = \lambda_n$ (resp. $\kappa_{|s|} > \lambda_n$). We claim that $C = \{\gamma < \lambda_n : \forall U \text{ cobounded below } \gamma \exists B \in [T_n] \cap [U](\mathrm{cl}_f \operatorname{ran} B \cap \lambda_n \subset \gamma)\}$ contains a club set.

Suppose to the contrary that we have a stationary set $S \subset \lambda$ and for $\gamma \in S$ a subtree U_γ of W cobounded below γ with $\mathrm{cl}_f \operatorname{ran} B \cap \lambda_n \not\subset \gamma$ for any $B \in [T_n] \cap [U_\gamma]$. Build inductively a subtree T of T_n so that $\mathrm{suc}_T(s)$ is $\mathrm{suc}_{T_n}(s)$ if $\kappa_{|s|} \leq \lambda_n$, and is $\{\alpha\}$ with $s * \langle\alpha\rangle \in \bigcap\{U_\gamma : s \in U_\gamma\}$ otherwise. Note that the map $s \mapsto s|\{i : \kappa_i = \lambda_n\}$ is injective on $\{s \in T : \kappa_{|s|} = \lambda_n\}$. Hence $D = \{\gamma < \lambda_n : \forall s \in T((\kappa_{|s|} = \lambda_n \wedge s``\{i : \kappa_i = \lambda_n\} \subset \gamma) \Rightarrow (\mathrm{cl}_f \operatorname{ran} s \cap \lambda_n \subset \gamma \wedge \gamma \in \lim \mathrm{suc}_T(s)))\}$ contains a club set. Fix $\gamma \in S \cap D$. Take inductively $B \in [T] \cap [U_\gamma]$ as follows: Suppose that $s \in T \cap U_\gamma$ is defined. Then $\mathrm{suc}_T(s) \cap \mathrm{suc}_{U_\gamma}(s) \neq \emptyset$, since $\mathrm{suc}_{U_\gamma}(s) = \kappa_{|s|}$ when $\kappa_{|s|} < \lambda_n$, since $\mathrm{suc}_{U_\gamma}(s)$ is cobounded in γ and $\mathrm{suc}_T(s)$ is unbounded in γ by $\gamma \in D$, $s \in T$ and $s``\{i : \kappa_i = \lambda_n\} \subset \gamma$ when $\kappa_{|s|} = \lambda_n$, and by $s \in U_\gamma$ and the choice of $\mathrm{suc}_T(s)$ when $\kappa_{|s|} > \lambda_n$. Thus $\mathrm{cl}_f \operatorname{ran} B \cap \lambda_n = \bigcup\{\mathrm{cl}_f B``i \cap \lambda_n : \kappa_i = \lambda_n\} \subset \gamma$ by $\gamma \in D$ and $B|i \in T$. This contradicts $\mathrm{cl}_f \operatorname{ran} B \cap \lambda_n \not\subset \gamma$ by $\gamma \in S$ and the choice of U_γ.

Fix $\gamma \in S_n \cap C$. Set $T^* = \{s \in T_n : \forall t \leq s \forall U \ni t \text{ cobounded below } \gamma \exists B \in [T_n] \cap [U](t \subset B \wedge \mathrm{cl}_f \operatorname{ran} B \cap \lambda_n \subset \gamma)\}$, a subtree of T_n. Note that $\emptyset \in T^*$ by $\gamma \in C$. Fix $s \in T^*$. We claim that $\mathrm{suc}_{T^*}(s)$ is a singleton if $\kappa_{|s|} < \lambda_n$, and is unbounded in γ (resp. $\kappa_{|s|}$) if $\kappa_{|s|} = \lambda_n$ (resp. $\kappa_{|s|} > \lambda_n$). We show the case $\kappa_{|s|} = \lambda_n$. The case $\kappa_{|s|} > \lambda_n$ (resp. $\kappa_{|s|} < \lambda_n$) is given by a similar (resp. simpler) argument.

Suppose to the contrary that $A = \gamma - \mathrm{suc}_{T^*}(s)$ is cobounded. Then for $\alpha \in A$ we have a subtree $U_\alpha \ni s*\langle\alpha\rangle$ of W cobounded below γ such that $\mathrm{cl}_f \operatorname{ran} B \cap \lambda_n \not\subset \gamma$ for any $s * \langle\alpha\rangle \subset B \in [T_n] \cap [U_\alpha]$ by $s \in T^*$ and $s * \langle\alpha\rangle \notin T^*$. Fix a subtree U of W cobounded below γ with $\{t \in U : s < t\} = \bigcup_{\alpha \in A}\{t \in U_\alpha : s * \langle\alpha\rangle \leq t\}$. Take $s \subset B \in [T_n] \cap [U]$ with $\mathrm{cl}_f \operatorname{ran} B \cap \lambda_n \subset \gamma$ by $s \in T^*$, and then $\alpha \in A$ with $s*\langle\alpha\rangle \subset B \in [U_\alpha]$ by the minimal choice of U. This contradicts $\mathrm{cl}_f \operatorname{ran} B \cap \lambda_n \not\subset \gamma$ by $s * \langle\alpha\rangle \subset B \in [T_n] \cap [U_\alpha]$ and the choice of U_α.

Now fix an unbounded set $\{\gamma_i : i < \omega\} \subset \gamma$. Build inductively a subtree T_{n+1} of T^* so that $\mathrm{suc}_{T_{n+1}}(s)$ is $\mathrm{suc}_{T^*}(s)$ if $\kappa_{|s|} \neq \lambda_n$, and is $\{\alpha\}$ with $\gamma_m < \alpha < \gamma$ otherwise, where $m = |\{i < |s| : \kappa_i = \lambda_n\}|$. Then T_{n+1} is as desired: Fix $B \in [T_{n+1}]$. Then $\sup(\mathrm{cl}_f \operatorname{ran} B \cap \lambda_n) = \gamma$, since $\sup\{B(i) : \kappa_i = \lambda_n\} = \gamma$ and $\mathrm{cl}_f \operatorname{ran} B \cap \lambda_n = \bigcup_{i<\omega} \mathrm{cl}_f B``i \cap \lambda_n \subset \gamma$ by $B|i \in T^*$.

Finally $\bigcap_{n<\omega} T_n$ has a unique branch B and $\sup(\mathrm{cl}_f \operatorname{ran} B \cap \lambda_n) \in S_n$ for any

$n < \omega$ as desired. ∎

We are ready to prove the main result of this paper:

THEOREM 1: *$\mathcal{P}_\kappa\lambda$ splits into λ^ω stationary sets.*

Proof: When $\lambda \le \mu^\omega$ for some regular cardinal $\kappa < \mu \le \lambda$, fix a club guessing sequence $\langle c_\gamma : \gamma \in S^\omega_\mu\rangle$ with $c_\gamma = \{\gamma_n : n < \omega\}$ and split S^κ_μ into stationary sets $\{S_\xi : \xi < \mu\}$. Then for $p : \omega \to \mu$, $\{x \in \mathcal{P}_\kappa\lambda : \exists\gamma \in S^\omega_\mu(\sup(x \cap \mu) = \gamma \wedge \forall n < \omega(\min(x - \gamma_n) \in S_{p(n)}))\}$ is stationary by Main Lemma 1, and mutually disjoint.

When $\operatorname{cf}\lambda = \omega$, fix an unbounded set $\{\lambda_n : n < \omega\} \subset \lambda$ of regular cardinals $> \kappa$. Then $|\prod_{n<\omega}\lambda_n| = \lambda^\omega$. For $n < \omega$ split $S^\omega_{\lambda_n}$ into stationary sets $\{S_{n\xi} : \xi < \lambda_n\}$. Then for $p \in \prod_{n<\omega}\lambda_n$, $\{x \in \mathcal{P}_\kappa\lambda : \forall n < \omega(\sup(x \cap \lambda_n) \in S_{np(n)})\}$ is stationary by Main Lemma 2, and mutually disjoint.

Otherwise we have $\omega < \operatorname{cf}\lambda < \lambda$ and $\alpha^\omega < \lambda$ for any $\alpha < \lambda$, and hence $\lambda^\omega = \lambda$. For completeness we provide a proof implicit in [T1]. First we claim that $\{x \in \mathcal{P}_\kappa\lambda : \sup(x \cap \mu) \in S \wedge \sup(x \cap \nu) \in S'\}$ is stationary for any regular cardinals $\kappa \le \mu < \nu < \lambda$ and stationary sets $S \subset S^\omega_\mu$ and $S' \subset S^\omega_\nu$. Fix $f\colon \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$. Take $\beta \in S'$ with $\operatorname{cl}_f\beta \cap \nu = \beta$, and an unbounded set $b \subset \beta$ of size ω, and then $\alpha \in S$ with $\operatorname{cl}_f(\alpha \cup b) \cap \mu = \alpha$, and an unbounded set $a \subset \alpha$ of size ω. Then $\sup(\operatorname{cl}_f(a \cup b) \cap \mu) = \alpha$ and $\sup(\operatorname{cl}_f(a \cup b) \cap \nu) = \beta$ as desired. Now set $\mu = \max\{\kappa, \operatorname{cf}\lambda\} < \lambda$ and split S^ω_μ into stationary sets $\{S_\xi : \xi < \operatorname{cf}\lambda\}$. Also fix an unbounded set $\{\lambda_\xi : \xi < \operatorname{cf}\lambda\} \subset \lambda$ of regular cardinals $> \mu$ and for $\xi < \operatorname{cf}\lambda$ split $S^\omega_{\lambda_\xi}$ into stationary sets $\{S_{\xi\zeta} : \zeta < \lambda_\xi\}$. Then for $(\xi\zeta) \in \sum_{\xi<\operatorname{cf}\lambda}\lambda_\xi$, $\{x \in \mathcal{P}_\kappa\lambda : \sup(x \cap \mu) \in S_\xi \wedge \sup(x \cap \lambda_\xi) \in S_{\xi\zeta}\}$ is stationary by the claim above, and mutually disjoint. ∎

Our second result is inspired by Burke's theorem [BMat] that the club filter on $\mathcal{P}_\kappa\lambda$ is not λ^+-saturated when $\kappa > \omega_1$ and $\operatorname{cf}\lambda < \kappa$:

THEOREM 2: *$\mathcal{P}_\kappa\lambda$ splits into λ^+ stationary sets when $\operatorname{cf}\lambda < \kappa$.*

Proof: The case $\operatorname{cf}\lambda = \omega$ follows from Theorem 1.

Otherwise fix a scale $\{f_\gamma : \gamma < \lambda^+\} \subset \prod_{\xi<\operatorname{cf}\lambda}\lambda_\xi$ with $\lambda_0 > \kappa$. Define $\rho\colon \mathcal{P}_\kappa\lambda \to \lambda^+$ by $\rho(x) = \min\{\gamma < \lambda^+ : \langle\sup(x \cap \lambda_\xi) : \xi < \operatorname{cf}\lambda\rangle \le^* f_\gamma\}$. We show that $\rho^{-1}S$ is stationary in $\mathcal{P}_\kappa\lambda$ for any stationary set $S \subset S^\omega_{\lambda^+}$.

Fix a club set $C \subset \mathcal{P}_\kappa\lambda$. Construct $\{x_a : a \in [\lambda^+]^{<\omega}\} \subset C$ by induction on $|a|$ so that $\operatorname{ran} f_{\max a} \subset x_a \subset x_b$ for any $a \subset b \in [\lambda^+]^{<\omega}$ by $\operatorname{cf}\lambda < \kappa$. Take $\gamma \in S$ with $\rho(x_a) < \gamma$ for any $a \in [\gamma]^{<\omega}$, and an unbounded set $B \subset \gamma$ of order type ω. Set $x = \bigcup_{\beta\in B} x_{B\cap\beta} \in C$. We claim that $\rho(x) = \gamma$ as desired.

First we have $\rho(x) \geq \gamma$, since for any $\beta \in B$, $\rho(x) \geq \rho(x_{B\cap\beta}) \geq \max(B \cap \beta)$ by $\operatorname{ran} f_{\max(B\cap\beta)} \subset x_{B\cap\beta}$. Next

$$\langle \sup(x \cap \lambda_\xi) : \xi < \operatorname{cf} \lambda \rangle = \langle \sup_{\beta \in B} \sup(x_{B\cap\beta} \cap \lambda_\xi) : \xi < \operatorname{cf} \lambda \rangle \leq^* f_\gamma,$$

since $\operatorname{cf} \lambda > \omega$ and for any $\beta \in B$, $\langle \sup(x_{B\cap\beta} \cap \lambda_\xi) : \xi < \operatorname{cf} \lambda \rangle \leq^* f_\gamma$ by $\rho(x_{B\cap\beta}) < \gamma$.

Now split $S^\omega_{\lambda^+}$ into stationary sets $\{S_\alpha : \alpha < \lambda^+\}$. Then for $\alpha < \lambda^+$, $\rho^{-1}S_\alpha$ is stationary in $\mathcal{P}_\kappa\lambda$ by the claim above, and mutually disjoint. ∎

3. Some remarks

For the moment let us assume that $\mu < \kappa < \lambda$ are all regular and consider the stationary set $S^\mu_{\kappa\lambda} = \{x \in \mathcal{P}_\kappa\lambda : \operatorname{cf} \sup x = \mu\}$. Main Lemma 1 implies that $S^\omega_{\kappa\lambda}$ splits into λ^ω stationary sets. On the other hand Matsubara [Mat] proved that a stationary subset of $S^\mu_{\kappa\lambda}$ splits into λ stationary sets. This is optimal when $\mu > \omega$ and $\lambda < \kappa^{+\omega}$, since Baumgartner [B] shows that $|\{x \in \mathcal{P}_\kappa\lambda : \kappa \leq \forall\nu \leq \lambda(\operatorname{cf} \sup(x \cap \nu) > \omega)\} \cap C| = \lambda$ for some club set $C \subset \mathcal{P}_\kappa\lambda$. In fact the map $x \mapsto \langle \sup(x \cap \nu) : \kappa \leq \nu \leq \lambda \rangle$ is injective on this set. Complementing a result of Abe [A], we remark that the map $x \mapsto \sup x$ is not injective on $S^\mu_{\kappa\lambda} \cap C$ for any club set $C \subset \mathcal{P}_\kappa\lambda$: Fix $f : \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$ generating C. Take $\kappa < \gamma \in S^\mu_\lambda$ closed under f, an unbounded set $a \subset \gamma$ of size μ and $\alpha \in \gamma - \operatorname{cl}_f a$. Then $\operatorname{cl}_f a \neq \operatorname{cl}_f(a \cup \{\alpha\})$ and $\sup \operatorname{cl}_f a = \sup \operatorname{cl}_f(a \cup \{\alpha\}) = \gamma$ as desired.

The rest of the section is devoted to a detailed proof of the Donder–Matet theorem mentioned earlier.

Let $\mu > \omega$ be regular and $d_\gamma = \{\gamma_n : n < \omega\} \subset \gamma$ unbounded for $\gamma \in S^\omega_\mu$. The following lemma from [B] (see also [BT]), where it is stated in (harmlessly) inaccurate form, is implicit in Lemma 9.1 of [DM].

LEMMA 1: *Let $S \subset S^\omega_\mu$ be stationary. Then $\{\alpha < \mu : \{\gamma \in S : \alpha \in d_\gamma\}$ is stationary$\}$ is unbounded.*

Proof: Suppose to the contrary that we have $\beta < \mu$ and for $\beta < \alpha < \mu$ a club set $C_\alpha \subset \mu$ with $C_\alpha \cap \{\gamma \in S : \alpha \in d_\gamma\} = \emptyset$. Take $\beta < \gamma \in S \cap \Delta_{\beta<\alpha<\mu} C_\alpha$. Then for any $\beta < \alpha < \gamma$, $\alpha \notin d_\gamma$ by $\gamma \in S \cap C_\alpha$. This contradicts the unboundedness of d_γ in γ. ∎

We call a subtree $T \neq \emptyset$ of $[\mu]^{<\omega}$ in the sense of Section 2 unbounded (resp. cobounded) if $\operatorname{suc}_T(a)$ is unbounded (resp. cobounded) in μ for any $a \in T$. The following lemma from [RS] (see also [BMag]) would ensure that the map ξ in Lemma 9.2 of [DM] is well-defined (at least in the case we are interested in).

LEMMA 2: *Let $g: T \to \nu$ with T an unbounded subtree of $[\mu]^{<\omega}$ and $\nu^\omega < \mu$. Then for some unbounded subtree T^* of T, g is constant on $T^* \cap [\mu]^n$ for any $n < \omega$.*

Proof: For $h: \omega \to \nu$ set $T_h = \{a \in T: \forall b \leq a(g(b) = h(|b|))\}$, a subtree of T. First we find $h : \omega \to \nu$ with $[T_h] \cap [U] \neq \emptyset$ for any cobounded subtree U of $[\mu]^{<\omega}$.

Suppose to the contrary that for $h : \omega \to \nu$ we have a cobounded subtree U_h of $[\mu]^{<\omega}$ with $[T_h] \cap [U_h] = \emptyset$. Take inductively $B \in [T] \cap [\bigcap\{U_h : h : \omega \to \nu\}]$ by $\nu^\omega < \mu$. Take $h: \omega \to \nu$ with $B \in [T_h]$. This contradicts $[T_h] \cap [U_h] = \emptyset$.

Now fix $h: \omega \to \nu$ as above. Set $T^* = \{a \in T_h : \forall b \leq a \forall U \ni b$ cobounded $\exists B \in [T_h] \cap [U](b \subset B)\}$, a subtree of T. Note that $\emptyset \in T^*$ by the choice of h. We claim that T^* is unbounded as desired.

Suppose to the contrary that $A = \mu - \mathrm{suc}_{T^*}(a)$ is cobounded for some $a \in T^*$. Then for $\alpha \in A$ we have a cobounded subtree $U_\alpha \ni a \cup \{\alpha\}$ of $[\mu]^{<\omega}$ such that $a \cup \{\alpha\} \not\subset B$ for any $B \in [T_h] \cap [U_\alpha]$ by $a \in T^*$ and $a \cup \{\alpha\} \notin T^*$. Fix a cobounded subtree U of $[\mu]^{<\omega}$ with $\{b \in U : a < b\} = \bigcup_{\alpha \in A}\{b \in U_\alpha : a \cup \{\alpha\} \leq b\}$. Take $a \subset B \in [T_h] \cap [U]$ by $a \in T^*$, and then $\alpha \in A$ with $a \cup \{\alpha\} \subset B \in [U_\alpha]$ by the minimal choice of U. This contradicts $a \cup \{\alpha\} \not\subset B$ by $B \in [T_h] \cap [U_\alpha]$ and the choice of U_α. ∎

We are ready to prove the main claim of Proposition 9.6 of [DM]:

THEOREM: *Let $\lambda > 2^{<\kappa}$. Then there is a sequence $\langle v_x : x \in \mathcal{P}_\kappa\lambda\rangle$ such that $\{x \in \mathcal{P}_\kappa\lambda : v_x = X \cap x\}$ is stationary for any $X \subset \lambda$.*

Proof: Set $\mu = (2^{<\kappa})^+$ and split S^ω_μ into stationary sets $\{S^w : w \in \mathcal{P}_\kappa\kappa\}$. For $x \in \mathcal{P}_\kappa\lambda$ with $\mathrm{cf}\sup(x \cap \mu) = \omega$ set $v_x = \pi(x)^{-1}w$, where $\sup(x \cap \mu) \in S^w$ and $\pi(x): x \to \mathrm{ot}\, x$ is the increasing bijection. Fix $X \subset \lambda$. We show that $\{x \in \mathcal{P}_\kappa\lambda : v_x = X \cap x\}$ is stationary.

Fix $f: \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$. We build inductively an unbounded subtree T of $[\mu]^{<\omega}$ and for $a \in T$ a stationary set $S_a \subset S^\omega_\mu$ and an increasing injection $\chi_a: \mathrm{cl}_f\, a \to \kappa$ so that for any $a \leq b \in T$, $S_b \subset S_a$ and for any $\gamma \in S_a$, $a \subset d_\gamma$ and $\pi(\mathrm{cl}_f\, d_\gamma)|\,\mathrm{cl}_f\, a = \chi_a$. Note that $\chi_a \subset \chi_b$ for any $a \leq b \in T$.

First set $S_\emptyset = S^\omega_\mu$ and $\chi_\emptyset = \emptyset$. Next suppose that $T \cap [\mu]^n$ and S_a for $a \in T \cap [\mu]^n$ are defined. Fix $a \in T \cap [\mu]^n$. Let

$$\mathrm{suc}_T(a) = \{\alpha < \mu : \max a < \alpha \wedge \{\gamma \in S_a : \alpha \in d_\gamma\} \text{ is stationary}\},$$

which is unbounded by Lemma 1. Fix $\alpha \in \mathrm{suc}_T(a)$. Take a stationary set $S_{a\cup\{\alpha\}} \subset \{\gamma \in S_a : \alpha \in d_\gamma\}$ and $\chi_{a\cup\{\alpha\}} : \mathrm{cl}_f(a \cup \{\alpha\}) \to \kappa$ so that for any $\gamma \in S_{a\cup\{\alpha\}}$, $\pi(\mathrm{cl}_f\, d_\gamma)|\,\mathrm{cl}_f(a \cup \{\alpha\}) = \chi_{a\cup\{\alpha\}}$ by $2^{<\kappa} < \mu$.

By Lemma 2 with $\nu = 2^{<\kappa}$ take an unbounded subtree T^* of T and $\{y_n : n < \omega\}$, $\{z_n : n < \omega\} \subset \mathcal{P}_\kappa\kappa$ so that $\operatorname{ran}\chi_a = y_n$ and $\chi_a``(X \cap \mathrm{cl}_f\, a) = z_n$ for any $a \in T^* \cap [\mu]^n$. Then

$$C = \{\gamma < \mu : \mathrm{cl}_f\, \gamma \cap \mu = \gamma \wedge \forall a \in T^* \cap [\gamma]^{<\omega}(\gamma \in \lim \mathrm{suc}_{T^*}(a))\}$$

contains a club set. Set $w = \pi(\bigcup_{n<\omega} y_n)``\bigcup_{n<\omega} z_n \in \mathcal{P}_\kappa\kappa$. Fix $\gamma \in S^w \cap C$. Take inductively $B = \{\beta_n : n < \omega\} \in [T^*]$ so that $\gamma_n < \beta_n < \gamma$ by $\gamma \in C$ and the inductive hypothesis $\{\beta_i : i < n\} \in T^* \cap [\gamma]^{<\omega}$. Then $\mathrm{cl}_f\, B$ is as desired: First we have $\sup(\mathrm{cl}_f\, B \cap \mu) = \gamma$, since $\sup B = \gamma$ and $\mathrm{cl}_f\, B \cap \mu \subset \mathrm{cl}_f\, \gamma \cap \mu = \gamma$ by $\gamma \in C$. Next $\pi(\mathrm{cl}_f\, B)``(X \cap \mathrm{cl}_f\, B) = w$, since $\chi = \bigcup_{\beta \in B} \chi_{B \cap \beta}$: $\mathrm{cl}_f\, B \to \bigcup_{n<\omega} y_n$ is an increasing bijection and $\chi``(X \cap \mathrm{cl}_f\, B) = \bigcup_{n<\omega} z_n$ by the note above. ∎

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